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Self-similar, time-dependent flows with a free surface

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A simple derivation is given of the parabolic flow first described by John (1953) in semi-Lagrangian form. It is shown that the scale of the flow decreases like t^{-3}, and the free surface contracts about a point which lies one-third of the way from the vertex of the parabola to the focus.

The flow is an exact limiting form of either a Dirichlet ellipse or hyperbola, as the time t tends to infinity.

Two other self-similar flows, in three dimensions, are derived. In one, the free surface is a paraboloid of revolution, which contracts like t^{-2} about a point lying one-quarter the distance from the vertex to the focus. In the other, the flow is non-axisymmetric, and the free surface contracts like t^{-5}.

The parabolic flow is shown to be one of a general class of self-similar flows in the plane, described by rational functions of degree n. The parabola corresponds to $n = 2$. When $n = 3$ there are two new flows. In one of these the scale varies as $t^{\frac{12}{7}}$ and the free surface has the appearance of a trough filling up. In the other, the free surface resembles flow round the end of a rigid wall; the scale varies as $t^{-4\cdot17}$.

1. Introduction

The review of free-surface flows by Gilbarg (1960) convincingly emphasizes the scarcity of known, exact solutions to time-dependent flows with a free surface, particularly when gravitational terms are included. Among the known solutions are the accelerated cavities of von Kármán (1949) and Gilbarg (1952), which have also been extended by Yih (1960); the similarity flows for the impact of a cone on a free surface (Garabedian 1953), and some interesting examples of free-surface flows derived by an inverse method due to John (1953). Not mentioned by Gilbarg (1960) are the long-standing exact solutions known as Dirichlet ellipsoids (Lamb 1932, p. 382), which were first derived by Dirichlet in 1860, and partly rediscovered by John (1952) and Taylor (1960). A hyperbolic form of this solution was recently discussed by Longuet-Higgins (1972) in relation to the appearance of instabilities at the crest of a standing wave.

Among the very simplest of non-trivial solutions to the time-dependent problem is the flow derived in §2 below. In this, the free surface takes the form of a parabola, whose linear dimensions vary as t^{-3} (where t denotes the time) and which therefore reduces to a thin sheet as $t \to \infty$. This flow was in fact discovered by John (1953), in whose treatment, however, the nature of the solution is somewhat hidden by the inclusion of gravity in a non-essential way (see § 7, below). When viewed in the natural frame of reference it becomes clear that the flow is self-similar, and that the parabolic surface contracts (or expands) about a fixed point one-third of the distance from the vertex to the focus.

The purpose of the present paper is to investigate the possibility of other flows of a similar kind. In § 3 we derive two flows in three dimensions, in which the free surface contracts like t^{-2} and t^{-5} respectively. The possible connexion of all these flows with the Dirichlet ellipsoids is discussed in § 4, and in §§ 5 and 6 it is shown that the parabolic flow of § 2 is in fact an exact asymptotic form of the two-dimensional Dirichlet hyperbola. The analysis is given in some detail because of a possible future application to the theory of slender breaking waves (Longuet-Higgins & Cokelet 1976).

In the second part of the paper (§§ 7–9) the parabolic solution is generalized in another direction, namely to higher-order rational flows in a plane. For this purpose, the semi-Lagrangian formulation of John (1953) proves particularly useful. We show that for each positive integer n there exist two classes of self-similar, time-dependent flows, with a free parameter λ. By an appropriate choice of λ one can generally exclude certain singularities in the flow. The case $n = 2$ yields the aforementioned parabolic flow and another self-similar flow confined to the outside of a parabolic surface. The case $n = 3$ gives rise to a solution representing a fluid filling up a trough in an otherwise plane surface, and another solution representing a free-surface flow round the end of a solid wall.

All the flows discussed in this paper are gravity-free. The paper is intended to prepare the ground for a future study of time-dependent flows incorporating gravity in an essential way.

2. A two-dimensional, free-surface flow

We shall first derive by a direct method the parabolic flow mentioned in the introduction.

Consider the velocity potential

$$\phi = \frac{1}{2t}(x^2-y^2)+P\frac{x}{t^\lambda}, \tag{2.1}$$

where (x, y) are rectangular co-ordinates, t is the time and P and λ are constants to be determined. Taking the density as unity, we find that the pressure p, from Bernoulli's equation, is given by

$$-p = \frac{y^2}{t^2}+(1-\lambda)P\frac{x}{t^{\lambda+1}}+\frac{P^2}{2t^{2\lambda}}+f, \tag{2.2}$$

where f is a function of the time only. Hence the rate of change of p following a fixed particle is given by

$$\frac{Dp}{Dt} = \frac{4y^2}{t^3}+\lambda(1-\lambda)P\frac{x}{t^{\lambda+2}}+(2\lambda-1)\frac{P^2}{t^{2\lambda+1}}-\frac{df}{dt}. \tag{2.3}$$

At the free surface both p and Dp/Dt must vanish. The vanishing of (2.2) and (2.3) will represent the same surface provided the coefficients of corresponding terms are in proportion. Hence, either $\lambda = 1$ or $\lambda = 4$. If $\lambda = 1$ then from (2.1) we may, by choice of a different frame of reference, take $P = 0$, so

$$\frac{df}{dt} = -\frac{4}{t}f, \qquad f = \frac{Q}{t^4}$$

say. The free surface is then $y^2 = -Q/t^2$, which represents two planes parallel to the x axis. Leaving aside this flow, which has been described and demonstrated experimentally by Longuet-Higgins (1972),† consider the case $\lambda = 4$. The terms in p and Dp/Dt dependent on t alone will then be in proportion to the terms in x provided that

$$\frac{df}{dt} + \frac{4}{t}f = \frac{5P^2}{t^9},$$

whence we have

$$f = -\frac{5}{4}\frac{P^2}{t^8} + \frac{Q}{t^4}, \tag{2.4}$$

where Q is an arbitrary constant. The velocity potential is now

$$\phi = \frac{1}{2t}(x^2 - y^2) + P\frac{x}{t^4} \tag{2.5}$$

and the free surface is

$$y^2 = 3P\frac{x}{t^3} + \frac{3}{4}\frac{P^2}{t^6} - \frac{Q}{t^2}, \tag{2.6}$$

where P and Q are both arbitrary constants.

To fix the ideas, let us suppose $P < 0$ and $t > 0$. Then setting

$$\alpha = -\tfrac{3}{4}P/t^3, \qquad c = Q/3P \tag{2.7}$$

the equation of the free surface becomes

$$y^2 = -4\alpha(x - \tfrac{1}{3}\alpha - ct). \tag{2.8}$$

When $c = 0$ this obviously represents a parabola, with vertex at the point $(\frac{1}{3}\alpha, 0)$ and distance α between vertex and focus (see figure 1).

As t increases, so α, and all the dimensions of the parabola, vary as t^{-3}. The free surface contracts towards the origin O as the centre of similitude. This point lies inside the parabola, at one-third of the distance from the vertex to the focus.

When t is small, the free surface (for bounded values of y) is almost plane. When on the other hand $t \to \infty$ the free surface becomes very elongated in the x direction, producing a thin jet of fluid, ejected to the left with velocity

$$\phi_x = x/t + P/t^4$$

(this quantity being negative when $x < \frac{4}{3}\alpha$). When t is large, $\phi_x \sim x/t$, so that a given particle travels leftwards with almost uniform velocity. At a fixed point, however, the velocity tends to zero.

Remarkably, (2.8) shows that for *any* value of Q, not necessarily zero, a free surface may be found on which the pressure is not only constant but constant

† Note that in equation (6.4) of that paper, the last term should be t^{-1}.

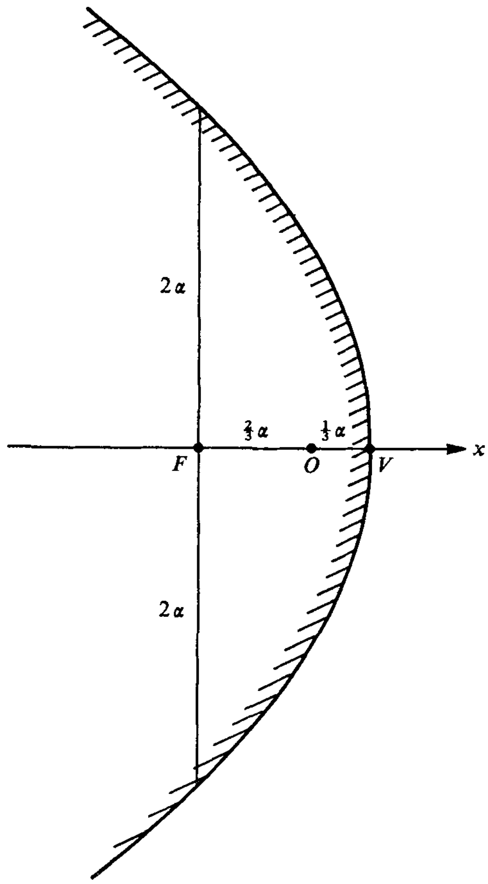

FIGURE 1. A cross-section of the free surface in the two-dimensional flow described in §2. The curve is a parabola which contracts about the point O, lying one-third of the way from V to F. The distance VF is proportional to t^{-3}.

following a particle. The surface always has the form of a parabola, parallel to the surface corresponding to $Q = 0$, but travelling to the right with constant velocity c. To produce this flow we simply reduce the pressure at infinity by the amount Q/t^4.

The velocity normal to the median plane of the jet is

$$\phi_y = -y/t,$$

which is always independent of x. It follows that any line of particles parallel to the median plane always remains so, and that the flow may be realized by the ejection of fluid from between two approaching parallel plates. A more interesting realization is likely to be in the jet of water formed by a 'plunging breaker' during the short time that the jet is thin and almost horizontal (see Longuet-Higgins & Cokelet 1976).

3. Self-similar flows in three dimensions

Let (x, y, z) be rectangular co-ordinates in three dimensions, and consider the more general potential

$$\phi = \frac{1}{2t}(x^2 - my^2 - nz^2) + P\frac{x}{t^\lambda}, \tag{3.1}$$

where in order to satisfy Laplace's equation we specify

$$m + n = 1. \tag{3.2}$$

As in §2 we have

$$-p = \frac{1}{2t^2}[m(m+1)y^2 + n(n+1)z^2] + (1-\lambda)\frac{Px}{t^{\lambda+1}} + \frac{P^2}{2t^{2\lambda}} + f, \tag{3.3}$$

where f is a function of t only, and

$$\frac{Dp}{Dt} = \frac{1}{t^3}[m(m+1)^2y^2 + n(n+1)^2z^2] + \lambda(1-\lambda)\frac{Px}{t^{\lambda+2}} + (2\lambda-1)\frac{P^2}{t^{2\lambda+1}} - \frac{df}{dt}. \tag{3.4}$$

On comparing coefficients of y^2 and z^2, we see that the surfaces $p = 0$ and $Dp/Dt = 0$ will coincide only if $(m, n) = (1, 0)$ or $(0, 1)$, which cases are equivalent to the two-dimensional flows discussed earlier; or if $(m, n) = (2, -1), (\frac{1}{2}, \frac{1}{2})$ or $(-1, 2)$. Taking first the case

$$(m, n) = (\tfrac{1}{2}, \tfrac{1}{2})$$

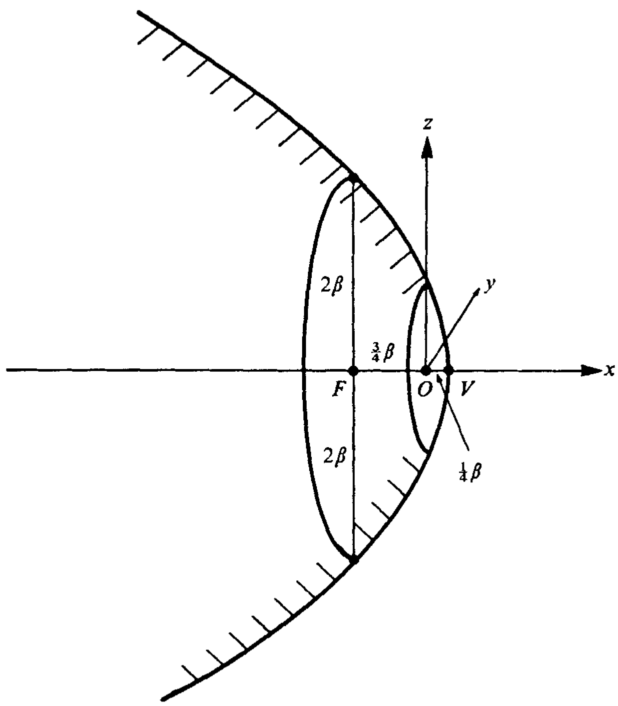

FIGURE 2. Sketch of the free surface in the axisymmetric flow given by (3.5). The surface contracts about the point O, lying one-quarter of the distance from V to the focus F. The dimensions vary as t^{-2}.

and comparing coefficients of y^2 and x we have

$$\lambda = 2(m+1) = 3,$$

and from the terms independent of x and y

$$f = -\frac{7}{6}\frac{P^2}{t^6} + \frac{Q}{t^3}.$$

Hence we obtain for the velocity potential

$$\phi = \frac{1}{2t}\left(x^2 - \frac{y^2+z^2}{2}\right) + P\frac{x}{t^3} \tag{3.5}$$

and for the free surface

$$y^2 + z^2 = -4\beta(x - \tfrac{1}{4}\beta - ct), \tag{3.6}$$

where

$$\beta = -\tfrac{4}{3}P/t^2, \qquad c = Q/2P. \tag{3.7}$$

Thus the free surface is a paraboloid of revolution (see figure 2), in which the dimensions vary as t^{-2}. The surface expands or contracts about a point lying *one-quarter* of the distance from the vertex to the focus.

If on the other hand we take $(m, n) = (-1, 2)$ we have

$$\lambda = 2(n+1) = 6$$

and

$$f = -\frac{4}{3}\frac{P^2}{t^{12}} + \frac{Q}{t^6}.$$

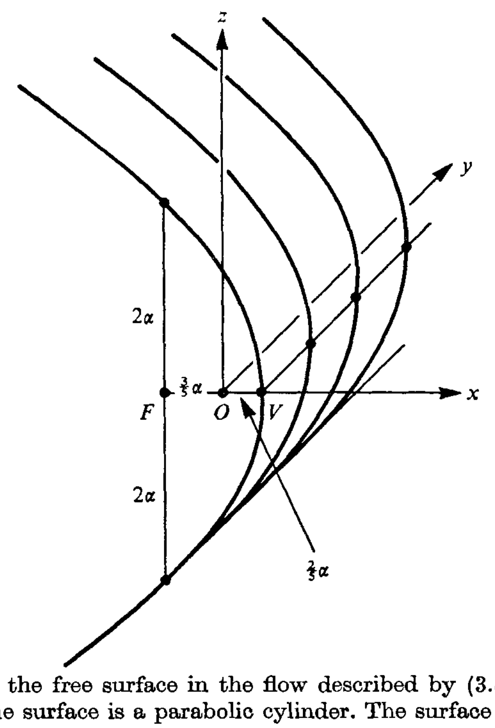

FIGURE 3. Sketch of the free surface in the flow described by (3.8). Though the flow is three-dimensional, the surface is a parabolic cylinder. The surface contracts about a line through O, lying two-fifths of the distance from the vertex to the focal line. The dimensions vary as t^{-6}.

Then the velocity potential is

$$\phi = \frac{1}{2t}(x^2+y^2-2z^2)+P\frac{x}{t^6} \tag{3.8}$$

and the free surface is

$$z^2 = -4\gamma(x-\tfrac{2}{5}\gamma-ct), \tag{3.9}$$

where $$\gamma = -\tfrac{5}{12}P/t^5, \qquad c = Q/5P. \tag{3.10}$$

Thus the flow is the sum of an axisymmetric flow and of a uniform translation in the x direction. The free surface is a parabolic cylinder (see figure 3) whose dimensions vary as t^{-5}. The surface expands or contracts about a line which lies $\frac{2}{5}$ of the distance between the vertex and the focal line.

4. The Dirichlet ellipsoids

It is natural to discuss the relation of the flows described in §§2 and 3 to the ellipsoids of Dirichlet (1860; see also Lamb 1932, §382). In these, the free surface is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1, \tag{4.1}$$

where a, b and c are functions of the time t only, satisfying

$$abc = \text{constant} = M \tag{4.2}$$

and, in the absence of gravitational attraction and vorticity,

$$a\ddot{a} = b\ddot{b} = c\ddot{c}, \tag{4.3}$$

where a dot denotes differentiation with respect to t. These last equations have the integral

$$\dot{a}^2+\dot{b}^2+\dot{c}^2 = \text{constant} = L^2 \tag{4.4}$$

and the corresponding velocity potential is

$$\phi = \frac{1}{2}\left(\frac{\dot{a}}{a}x^2+\frac{\dot{b}}{b}y^2+\frac{\dot{c}}{c}z^2\right). \tag{4.5}$$

In two cases the integrals (4.2) and (4.4) suffice to determine the motion, namely, first, when $\dot{c} \equiv 0$ and the motion is two-dimensional, in which case

$$b = M/ca, \quad \dot{b} = -M\dot{a}/ca^2$$

and so $$\dot{a} = L(1+M^2/c^2a^4)^{-\frac{1}{2}}; \tag{4.6}$$

and second, when $c \equiv b$ and the flow is axisymmetric, in which case

$$b^2 = M/a, \quad \dot{b}^2 = M\dot{a}^2/4a^3$$

and so $$\dot{a} = L(1+M/4a^3)^{-\frac{1}{2}}. \tag{4.7}$$

One might expect that the solutions found in §§2 and 3 were special cases of (4.6) or (4.7). This turns out to be untrue, but we can nevertheless show that they are *limiting* cases in a certain sense.

In the following sections we shall study a special two-dimensional case which in fact corresponds to the limit $c \to \infty$ with b^2 replaced by $-b^2$. This analysis is given fully, since it may have later application to the theory of breaking waves (Longuet-Higgins & Cokelet 1976).

5. The Dirichlet hyperbola

We shall now describe how the parabolic flow derived in § 2 is related to a class of exact, irrotational flows in which the free surface has the form of a variable hyperbola.

In the hyperbolic flow discussed by Longuet-Higgins (1972) the velocity potential is given by

$$\phi = \tfrac{1}{2}A(x^2 - y^2) \tag{5.1}$$

and the free surface by

$$-(\dot{A} + A^2)x^2 + (\dot{A} - A^2)y^2 = R^2A^4, \tag{5.2}$$

where A is a function of t only, defined by

$$t = \int_A^\infty \frac{dA}{A^2(1+N^4A^4)^{\frac{1}{2}}}, \tag{5.3}$$

R and N being any positive constants. It is convenient to write

$$AN = \alpha, \qquad t/N = \tau, \tag{5.4}$$

so that α and τ are related simply by

$$\tau = \int_\alpha^\infty \frac{d\alpha}{\alpha^2(1+\alpha^4)^{\frac{1}{2}}}. \tag{5.5}$$

Generally, when $\alpha > 0$, the substitution

$$\alpha = \cot(\tfrac{1}{2}\beta)$$

enables τ to be expressed as

$$\tau = \tan(\tfrac{1}{2}\beta)(1 - \tfrac{1}{2}\sin^2\beta)^{\frac{1}{2}} - E(\beta, 2^{-\frac{1}{2}}) + \tfrac{1}{2}F(\beta, 2^{-\frac{1}{2}}), \tag{5.6}$$

where E and F are elliptic integrals, and then α may be plotted against τ with β as a parameter (see figure 4 of Longuet-Higgins 1972).†

Certain aspects of the motion may be expressed very simply in terms of α. Thus in Lagrangian co-ordinates we have

$$\dot{x} = \phi_x = Ax, \qquad \dot{y} = \phi_y = -Ay$$

so $$x = x_0 \exp\left(\int_0^t A\,dt\right) = x_0F(\tau), \quad y = y_0 \exp\left(-\int_0^t A\,dt\right) = y_0/F(\tau),$$

where (x_0, y_0) denote the co-ordinates at time $t = 0$ and

$$F(\tau) = \exp\left(\int_0^t A\,dt\right) = \exp\left(\int_0^\tau \alpha\,d\tau\right). \tag{5.7}$$

† In equation (4.6) of that reference, the modulus of the elliptic integral should be $2^{-\frac{1}{2}}$ as in equation (5.6).

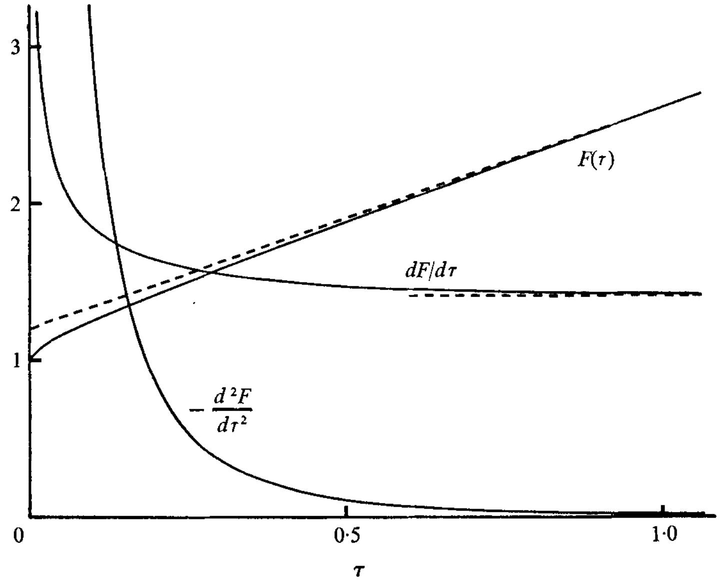

FIGURE 4. Graphs of the functions F, $dF/d\tau$ and $d^2F/d\tau^2$, which give the particle displacement, velocity and accelerations as functions of the time.

Since
$$\int_0^\tau \alpha\, d\tau = \int_\alpha^\infty \frac{d\alpha}{\alpha(1+\alpha^4)^{\frac{1}{2}}} = \tfrac{1}{2}\ln\left\{\frac{(1+\alpha^4)^{\frac{1}{2}}+1}{\alpha^2}\right\} \tag{5.8}$$
we have also
$$F = \{(1+\alpha^4)^{\frac{1}{2}}+1\}^{\frac{1}{2}}/\alpha. \tag{5.9}$$
The particle velocity $\dot{x}$ and acceleration $\ddot{x}$ are proportional respectively to $dF/d\tau$ and $d^2F/d\tau^2$. The quantities F, $dF/d\tau$ and $d^2F/d\tau^2$ are plotted in figure 4 above.

From (5.2), the free surface may be expressed as
$$x^2/a^2 - y^2/b^2 = 1, \tag{5.10}$$
where
$$\left.\begin{aligned} a &= \frac{R}{N}\frac{\alpha}{\{(1+\alpha^4)^{\frac{1}{2}}-1\}^{\frac{1}{2}}} = \frac{R}{N}F(\tau), \\ b &= \frac{R}{N}\frac{\alpha}{\{(1+\alpha^4)^{\frac{1}{2}}+1\}^{\frac{1}{2}}} = \frac{R}{N}\bigg/F(\tau). \end{aligned}\right\} \tag{5.11}$$
So the angle 2γ between the asymptotes is given by
$$\tan\gamma = b/a = F^{-2} \tag{5.12}$$
and we have
$$\tan 2\gamma = 2\tan\gamma/(1-\tan^2\gamma) = \alpha^2. \tag{5.13}$$
Lastly, the radius of curvature ρ at the vertex of the hyperbola is given by
$$\rho = b^2/a = (R/N)\,F^{-3}. \tag{5.14}$$
The angle 2γ is shown as a function of τ in figure 5.

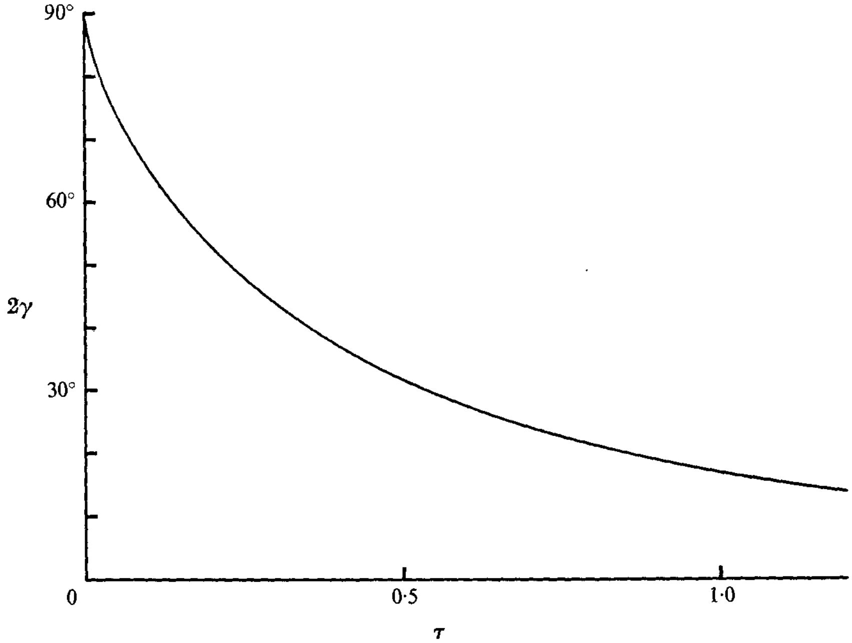

Figure 5. Graph of the angle 2γ between the asymptotes of the hyperbola, as a function of the dimensionless time τ.

When $\tau \to +0$ we have from (5.5)

$$\tau \sim \int_\alpha^\infty \frac{d\alpha}{\alpha^4} = \frac{1}{3\alpha^3}$$

so
$$\alpha \sim (3\tau)^{-\frac{1}{3}} \to \infty \tag{5.15}$$

and
$$F(\tau) \sim \exp\{\tfrac{1}{2}(3\tau)^{\frac{2}{3}}\}. \tag{5.16}$$

Hence
$$x \sim x_0\{1+\tfrac{1}{2}(3\tau)^{\frac{2}{3}}\}, \qquad N\dot{x} \sim x_0(3\tau)^{-\frac{1}{3}}, \qquad N^2\ddot{x} \sim -x_0(3\tau)^{-\frac{4}{3}} \tag{5.17}$$

and similarly for y. So as $\tau \to 0$ the displacements remain finite, though the velocities and accelerations become infinite. At $\tau = 0$ itself (5.12) and (5.13) show that the angle between the asymptotes is 90°, and from (5.14) the radius of curvature at the vertex equals R/N.

Hence we may define τ precisely, as the elapsed (dimensionless) time since the asymptotes were mutually perpendicular.

6. The form of the Dirichlet hyperbola as $t \to \infty$

Consider now the asymptotic form, as $t \to \infty$, of the solution given in § 5. When $\tau \to \infty$ then $\alpha \to 0$, and the appropriate expansion of the integral (5.5) is

$$\tau = \Delta + \frac{1}{\alpha} + \frac{1}{2}\frac{\alpha^3}{3} - \frac{1\,.\,3}{1\,.\,2}\frac{\alpha^7}{2^2\,.\,7} + \frac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3}\frac{\alpha^{11}}{2^3\,.\,11} - \dots, \tag{6.1}$$

where
$$\Delta = F(\tfrac{1}{2}\pi, 2^{-\frac{1}{2}}) - 2E(\tfrac{1}{2}\pi, 2^{-\frac{1}{2}}) = -0{\cdot}847213. \tag{6.2}$$

Writing
$$\tau - \Delta = \sigma \tag{6.3}$$
we have
$$\alpha^{-1} = \sigma - \tfrac{1}{6}\alpha^3 + \tfrac{3}{56}\alpha^7 - \dots, \tag{6.4}$$
so by successive approximation
$$\alpha^{-1} = \sigma - \tfrac{1}{6}\sigma^{-3} + \dots \tag{6.5}$$
and from (5.9)
$$F(\tau) = 2^{\frac{1}{2}}(\sigma - \tfrac{1}{24}\sigma^{-3} + \dots). \tag{6.6}$$

It will be seen from figure 4 that the linear term already gives a good approximation when $\tau > 0{\cdot}5$.

Consider the form of the free surface as $\tau \to \infty$. Then (5.11) and (6.6) show that
$$a \sim 2^{\frac{1}{2}}\sigma R/N = a_1, \qquad b \sim (2^{\frac{1}{2}}\sigma)^{-1} R/N = b_1 \tag{6.7}$$
say, so that the hyperbola becomes elongated in the x direction and correspondingly thin in the y direction. Moreover the vertex $(-a, 0)$ travels to the left with almost uniform velocity
$$-da_1/dt = -2^{\frac{1}{2}}R/N^2. \tag{6.8}$$

More accurately we have
$$a = a_1 + a_2 + \dots, \qquad b = b_1 + b_2 + \dots, \tag{6.9}$$
where
$$a_2 = -\frac{R}{N}\frac{\surd 2}{24}\sigma^{-3}, \qquad b_2 = \frac{R}{N}\frac{1}{24\surd 2}\sigma^{-5} \tag{6.10}$$
and in general a_n/a_{n-1} and b_n/b_{n-1} are of order σ^{-4}.

Transforming to new co-ordinates
$$x' = x + a_1, \qquad y' = y \tag{6.11}$$
corresponding to an observer moving to the left with *uniform* velocity $-da_1/dt$, and setting $x = x' - a_1$ in (5.10), we find for the new equation of the free surface
$$\frac{(x'-a_1)^2}{a^2} - \frac{y^2}{b^2} = 1$$
or
$$y^2 = (b^2/a^2)\,[(x'-a_1)^2 - (a_1 + a_2 + \dots)^2] \tag{6.12}$$
precisely. If now x' is of order $\sigma^{-4}a_1$, we have to a first approximation
$$y^2 = (-b_1^2/a_1^2)\,(2a_1 x' + 2a_1 a_2). \tag{6.13}$$

But writing
$$\alpha = \frac{b_1^2}{2a_1} = \frac{R}{N}\frac{1}{4\surd 2}\sigma^{-3} \tag{6.14}$$
we have from (6.10)
$$a_2 = -\tfrac{1}{3}\alpha, \tag{6.15}$$
so that (6.13) reduces to
$$y^2 = -4\alpha(x' - \tfrac{1}{3}\alpha), \tag{6.16}$$
which is of the same form as (2.8).

It follows that the self-similar flow described in § 2 is a limiting form of the flow near the tip of a Dirichlet hyperbola, as the time t tends to infinity.

It is interesting that the parabolic described in §2, though only an asymptotic form of the hyperbolic flow, is nevertheless an exact solution in itself. Higher approximations to the hyperbolic flow, for large values of τ, can easily be derived by including higher-order terms in (6.12). But the corresponding solutions are not exact, being valid only asymptotically for large τ. In the following section of this paper we shall describe a class of rational solutions which is indeed exact, and which effectively generalizes the solution described in §2.

7. A semi-Lagrangian method

John (1953) has given a general method for finding two-dimensional, time-dependent, irrotational flows with a free surface, as follows. Set $x+iy = \zeta$, and let us seek solutions in the form

$$\zeta = \zeta(\omega, t), \tag{7.1}$$

where ω is a Lagrangian co-ordinate (constant following a given particle) and ζ is an analytic function of ω. Thus the particle velocity and acceleration are respectively ζ_t and ζ_{tt}. The pressure gradient is in the direction of the vector $(\zeta_{tt}+ig)$, where g denotes gravity.

On the free surface ω is assumed to be real, so the tangent is in the direction of the vector ζ_ω. Hence the free-surface condition may be written as

$$\zeta_{tt}+ig = ir(\omega,t)\,\zeta_\omega, \qquad \omega \text{ real}, \tag{7.2}$$

where r is a function which is real on the boundary. The velocity potential $\chi = \phi+i\psi$ has to satisfy

$$d\chi/d\zeta = u-iv = [\zeta_t(\omega^*,t)]^* \tag{7.3}$$

on the boundary, which may be done by taking

$$\chi = \int [\zeta_t(\omega^*,t)]^*\,\zeta_\omega(\omega,t)\,d\omega$$

throughout the fluid. However $d\chi/d\zeta$ must be a single-valued analytic function of ζ throughout the fluid. Hence we have the restriction

$$[\zeta_t(\omega^*,t)]^* \text{ is a single-valued analytic function of } \zeta(\omega,t). \tag{7.4}$$

John (1953, p. 503) shows that all these conditions are satisfied by the expressions

$$\left.\begin{aligned}\zeta &= -it\omega^2 - iC\frac{\omega}{t} + i\left(\frac{1}{12}\frac{C^2}{t^3}+\tfrac{1}{2}gt^2\right),\\ \chi &= -\frac{\zeta^2}{2t}+i\left(\frac{C^2}{3t^4}+\tfrac{1}{2}gt\right)\zeta,\end{aligned}\right\} \tag{7.5}$$

where C is an arbitrary constant. We remark, first, that if the motion is referred to a frame of reference accelerating downwards with velocity g the terms in g disappear. The solution is therefore essentially gravity-free. Second, setting $g = 0$ in (7.5) we have

$$\phi = \mathscr{R}(\chi) = \frac{1}{2t}(y^2-x^2) - \tfrac{1}{3}C^2\frac{y}{t^4}, \tag{7.6}$$

which is identical with (2.5) if we rotate the axes through 90° and take

$$C^2 = -3P.$$

8. A general class of rational flows

Consider the polynomial expression

$$\zeta = c_0\omega^n + ic_1\frac{\omega^{n-1}}{t^\lambda} - c_2\frac{\omega^{n-2}}{t^{2\lambda}} - ic_3\frac{\omega^{n-3}}{t^{3\lambda}} + \ldots, \tag{8.1}$$

where n is any positive integer, and $c_0, \ldots, c_n$ and λ are constants to be determined. On substituting into (7.2), taking $g = 0$ and r independent of ω, and equating coefficients of $\omega^{n-1}, \omega^{n-2}, \ldots$, we obtain

$$\left.\begin{aligned} \lambda(\lambda+1)\,c_1 &= nc_0 rt^{\lambda+2}, \\ 2\lambda(2\lambda+1)\,c_2 &= (n-1)\,c_1 rt^{\lambda+2}, \\ 3\lambda(3\lambda+1)\,c_3 &= (n-2)\,c_2 rt^{\lambda+2}, \\ &\vdots \\ n\lambda(n\lambda+1)\,c_n &= 1\,.\,c_{n-1} rt^{\lambda+2}, \end{aligned}\right\} \tag{8.2}$$

which relations are satisfied by taking $r = \lambda/t^{\lambda+2}$, $c_0 = 1$ and

$$c_m = \frac{n!}{m!\,(n-m)!}\frac{1}{(\lambda+1)(2\lambda+1)\ldots(m\lambda+1)}, \qquad m = 1, \ldots, n, \tag{8.3}$$

provided that $\lambda \neq -1, -\frac{1}{2}, \ldots, -n^{-1}$. So, writing

$$t^{n\lambda}\zeta = X+iY, \qquad \lambda^{-1}t^{n\lambda+1}[\zeta_t(\omega^*, t)]^* = U-iV, \qquad t^\lambda\omega = \Omega, \tag{8.4}$$

we have from (8.1)

$$X+iY = Z(\Omega), \qquad U-iV = W(\Omega), \tag{8.5}$$

where

$$\left.\begin{aligned} Z &= \Omega^n + ic_1\Omega^{n-1} - c_2\Omega^{n-2} - ic_3\Omega^{n-3} + \ldots, \\ W &= \qquad ic_1\Omega^{n-1} + 2c_2\Omega^{n-2} - 3ic_3\Omega^{n-3} - \ldots. \end{aligned}\right\} \tag{8.6}$$

To avoid a singularity, any zero of $dZ/d\Omega$ in the domain of the fluid must also be a zero of $dW/d\Omega$. This gives us in general a condition to determine λ. Then the co-ordinates (x, y) and the velocity components (u, v) are given parametrically by

$$x+iy = t^{-n\lambda}Z, \quad u-iv = \lambda t^{-(n\lambda+1)}W. \tag{8.7}$$

In the special case $n = 2$ we have

$$Z = \Omega^2 + \frac{2i}{\lambda+1}\Omega - \frac{1}{(\lambda+1)(2\lambda+1)}, \tag{8.8}$$

$$W = \qquad \frac{2i}{\lambda+1}\Omega + \frac{2}{(\lambda+1)(2\lambda+1)}. \tag{8.9}$$

Now $dZ/d\Omega$ vanishes at $\Omega = -i/(\lambda+1)$, but since $dW/d\Omega$ has no zeros, the branch-point cannot be annulled. Nevertheless the solution given by (8.8) and (8.9) can, for general λ, still represent the flow *outside* a parabolic free surface, whose scale, from (8.4) varies like $t^{-2\lambda}$.

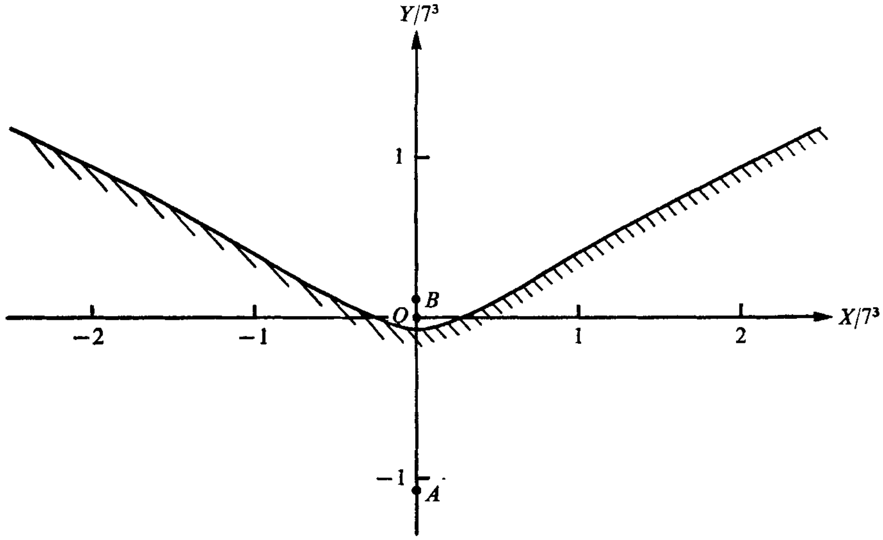

FIGURE 6. The form of the free surface in the self-similar flow described by (8.14). The points A and B are branch-points; A is annulled by choice of λ. The curve expands about the origin with dimensions proportional to $t^{\frac{12}{7}}$.

In the more typical case $n = 3$ we have

$$\left.\begin{aligned} Z &= \Omega^3+\frac{3i}{\lambda+1}\,\Omega^2-\frac{3}{(\lambda+1)(2\lambda+1)}\,\Omega-\frac{i}{(\lambda+1)(2\lambda+1)(3\lambda+1)},\\ W &= \qquad\frac{3i}{\lambda+1}\,\Omega^2+\frac{6}{(\lambda+1)(2\lambda+1)}\,\Omega-\frac{3i}{(\lambda+1)(2\lambda+1)(3\lambda+1)}.\end{aligned}\right\} \tag{8.10}$$

The derivatives of Z and W vanish when

$$\left.\begin{aligned} \Omega^2+\frac{2i}{\lambda+1}\,\Omega-\frac{1}{(\lambda+1)(2\lambda+1)} &= 0,\\ \frac{i}{\lambda+1}\,\Omega+\frac{1}{(\lambda+1)(2\lambda+1)} &= 0.\end{aligned}\right\} \tag{8.11}$$

Hence
$$\Omega = i/(2\lambda+1) \tag{8.12}$$

and on substituting in (8.11) we find

$$7\lambda+4 = 0,$$

so
$$\lambda = -\tfrac{4}{7}, \quad \Omega = -7i. \tag{8.13}$$

The Lagrangian co-ordinates are given by

$$X+iY = \Omega^3+7i\Omega^2+49\Omega-\frac{7^3}{15}\,i, \tag{8.14}$$

so the free surface, for which Ω is a real parameter, is given by

$$X = \Omega^3 + 49\Omega, \quad Y = 7\Omega^2 - \frac{7^3}{15}. \tag{8.15}$$

This is shown in figure 6. At large distances the tangent becomes nearly parallel to the X axis. From (8.11) the branch-points are given by

$$\Omega = -7i, \qquad Z = -\frac{16 \,.\, 7^3}{15} i \tag{8.16}$$

and

$$\Omega = \frac{7i}{3}, \qquad Z = \frac{16 \,.\, 7^3}{15 \,.\, 9} i. \tag{8.17}$$

The branch-point (8.16), which is on the $-Y$ axis, is already annulled, the other is not. So the solution is valid for a fluid filling the space on the convex side of the free surface. The motion is like an expanding trough. From (8.7), the dimensions of the trough vary like $t^{-n\lambda}$, that is like $t^{\frac{12}{7}}$. By reversing t we have a description of a trough 'filling up', though it must be remembered that the solution is essentially gravity-free.

9. A second class of flows

Consider now the alternative expression

$$\zeta = d_0 t\omega^n + id_1 \frac{\omega^{n-1}}{t^{\lambda-1}} - d_2 \frac{\omega^{n-2}}{t^{2\lambda-1}} - id_3 \frac{\omega^{n-3}}{t^{3\lambda-1}} + \dots. \tag{9.1}$$

Substituting into (7.2) and comparing coefficients as before we have

$$\left.\begin{aligned} (\lambda-1)\,\lambda d_1 &= nd_0 rt^{\lambda+2}, \\ (2\lambda-1)\,2\lambda d_2 &= (n-1)\,d_1 rt^{\lambda+2}, \\ (3\lambda-1)\,3\lambda d_3 &= (n-2)\,d_2 rt^{\lambda+2}, \\ &\vdots \\ (n\lambda-1)\,n\lambda d_n &= 1\,.\,d_{n-1} rt^{\lambda+2}, \end{aligned}\right\} \tag{9.2}$$

which are satisfied by taking $r = \lambda/t^{\lambda+2}$, $d_0 = 1$ and

$$d_m = \frac{n!}{m!\,(n-m)!}\frac{1}{(\lambda-1)(2\lambda-1)\dots(m\lambda-1)}, \qquad m = 1, \dots, n, \tag{9.3}$$

provided that $\lambda \neq 1, \frac{1}{2}, \dots, n^{-1}$. Now writing

$$t^{n\lambda-1}\zeta = X + iY, \qquad t^{n\lambda}[\zeta_t(\omega^*, t)]^* = U + iV, \qquad t^\lambda \omega = \Omega \tag{9.4}$$

we have

$$X + iY = Z(\Omega), \qquad U + iV = W(\Omega), \tag{9.5}$$

where now

$$\left.\begin{aligned} Z &= \Omega^n + id_1\,\Omega^{n-1} - d_2\,\Omega^{n-2} - id_3\,\Omega^{n-3} + \dots, \\ W &= \Omega^n + (\lambda-1)\,id_1\,\Omega^{n-1} + (2\lambda-1)\,d_2\,\Omega^{n-3} - \dots. \end{aligned}\right\} \tag{9.6}$$

We now proceed as before. In the case $n = 2$,

$$\left.\begin{aligned} Z &= \Omega^2+\frac{2i}{\lambda-1}\Omega-\frac{1}{(\lambda-1)(2\lambda-1)},\\ W &= \Omega^2+2i\Omega+\frac{1}{\lambda-1}. \end{aligned}\right\} \tag{9.7}$$

The only zero of $dZ/d\Omega$ is when $\Omega = -i/(\lambda-1)$, and on inserting this value in $dW/d\Omega = 0$ we get $\lambda-1 = 1$ so $\lambda = 2$. This gives the solution of §7.

In the next case, $n = 3$, we have

$$\left.\begin{aligned} Z &= \Omega^3+\frac{3i}{\lambda-1}\Omega^2-\frac{3}{(\lambda-1)(2\lambda-1)}\Omega-\frac{i}{(\lambda-1)(2\lambda-1)(3\lambda-1)},\\ W &= \Omega^3+3i\Omega^2+\frac{3}{\lambda-1}\Omega-\frac{i}{(\lambda-1)(2\lambda-1)} \end{aligned}\right\} \tag{9.8}$$

and so any root of

$$\Omega^2+\frac{2i}{\lambda-1}\Omega-\frac{1}{(\lambda-1)(2\lambda-1)} = 0 \tag{9.9}$$

lying within the domain of Ω must also be a root of

$$\Omega^2+2i\Omega+(\lambda-1)^{-1} = 0. \tag{9.10}$$

Subtracting we find

$$\Omega = i\lambda/(\lambda-2)(2\lambda-1) \tag{9.11}$$

and on substituting back into (9.9) we have

$$\lambda^2(\lambda-1)+2\lambda(\lambda-2)(2\lambda-1)+(\lambda-2)^2(2\lambda-1) = 0$$

or

$$7\lambda^3-20\lambda^2+16\lambda-4 = 0. \tag{9.12}$$

The only real root of this equation† is $\lambda = 1{\cdot}7231$, corresponding to $\Omega = -2{\cdot}544i$. So from (9.4) we have $\zeta = t^{-4\cdot169}Z$, where

$$Z = \Omega^3+4{\cdot}1488\Omega^2-1{\cdot}6960\Omega-0{\cdot}1356i. \tag{9.13}$$

Hence the free surface is given parametrically by

$$X = \Omega^3-1{\cdot}6970\Omega, \qquad Y = 4{\cdot}1488\Omega^2-0{\cdot}1356, \quad \Omega \text{ real} \tag{9.14}$$

(see figure 7). The branch-points are given by the roots of (9.9), which are $\Omega = -2{\cdot}544i$ or $-0{\cdot}2222i$, corresponding to $Z = -39{\cdot}14i$ or $0{\cdot}0254i$. The branch-point at $-39{\cdot}14i$ also corresponds to a root of W, so there is no singularity there. The branch-point at $0{\cdot}0254i$ is an irreducible branch-point of the flow, as we would expect from the fact that on this side of the origin the free surface intersects itself. The solution will, however, correspond to a physical flow if one sheet of the Riemann surface is excluded, say by a rigid boundary along the $-Y$ axis, as in figure 7. Physically, the solution represents the flow round the end of a solid

† The complex roots of (9.12) do not represent solutions, since λ would have to be replaced by λ^* in (9.10).

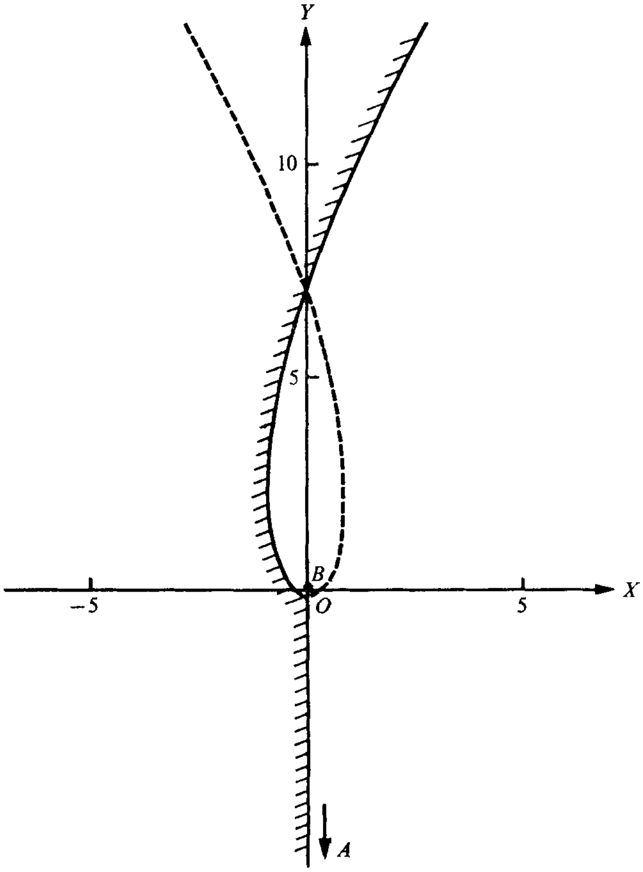

FIGURE 7. The form of the free surface in the self-similar flow described by (9.13). The branch-point B is excluded from the flow. The free surface contracts like $t^{-4 \cdot 17}$ and on the right becomes parallel to the X axis.

wall. As t increases to infinity the point of contact of the free surface moves out to the end of the wall, and the free surface curls tightly round it, becoming eventually almost plane and perpendicular to it.

Solutions for higher values of n may be investigated similarly.

It will be noticed that because the function $r(\omega, t)$ is the same in (8.2) and (9.2), and (7.2) is linear in ζ, any linear combination of (8.1) and (9.1) will satisfy (7.2) also, and indeed we may add expressions corresponding to different values of n. However, (8.4) and (9.4) show that the relation between ζ and Z is not the same in the two cases, or for different values of n, so that the corresponding expressions will not represent self-similar flows.

An exception might occur for two values of n, say n_1 and n_2, for which

$$n_1 \lambda = n_2 \lambda - 1, \qquad \lambda = 1/(n_2 - n_1).$$

But this possibility is excluded by the restrictions on λ which are implicit in (8.3) and (9.3).

Hence the two classes of flows given by (8.1) and (9.1) are possibly the only self-similar flows of polynomial type that can be readily realized. (We can of course replace ω by any rational function of ω alone, without essentially altering the flow.)

10. Discussion and conclusions

We have shown that the simple, parabolic flow described in § 2 may be generalized in two ways: first, to three dimensions, where two similar types of motion exist, both related to the Dirichlet ellipsoids. Just as the parabolic flow has been shown to be an exact, limiting form of the Dirichlet hyperbola, so it may be shown that the other two flows described in § 4 are exact limiting forms of axisymmetric hyperboloids having two and one sheets respectively.

The second generalization of the parabolic flow was to higher-order rational flows in the plane, using the semi-Lagrangian formalism of John. For each value of the positive integer n we found two possible flows. When $n = 2$, these are respectively the parabolic flow mentioned earlier, and another solution representing flow on the *outside* of a parabola. When $n = 3$, we obtained two new solutions, one representing a 'trough' expanding like $t^{\frac{12}{7}}$; the other, a flow curling round the end of a solid wall. Higher values of n will yield similar flows, generally restricted to part of the plane.

All the solutions described in the paper are essentially gravity-free, which may limit their application to highly accelerated flows, or to special situations where gravity can be neglected (see for example Longuet-Higgins 1972, § 8). To extend such solutions to situations where gravity is important, and particularly to the breaking of surface waves, is a problem to be considered in a subsequent paper.

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